B.sc(H) part1 paper 2
Topic:symmetric&skew-symmetric matrices
subject mathematatic
Dr Hari Kant Singh

Symmetric matrix : Definition : A square matrix $A = [a_{ij}]$ is said to be *symmetric* if $A = A'$ i.e., if $a_{ij} = a_{ji}$ i.e., the (i, j)th element is the same as the (j, i)th element.

Thus in a symmetric matrix $a_{ij} = a_{ji}$ for all i, j i.e., $a_{12} = a_{21}$, $a_{13} = a_{31}$, $a_{23} = a_{32}$, ...

For example, if $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then $A' = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

$\therefore \quad A = A'$

$\therefore$ A is symmetric.

Skew-symmetric matrices : Definition : A square matrix $A = [a_{ij}]$ is said to be *skew-symmetric* if $A = -A'$ i.e., if $a_{ij} = -a_{ji}$ i.e. the (i, j)th element is the negative of the (j, i)th element for all i, j.

Since, by definition $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \;\therefore\; a_{ii} = 0$.

Therefore the diagonal elements of skew-symmetric matrix are always zero.

For example, the matrix $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ is skew-symmetric.

Properties

I. The product of any matrix with its transpose is symmetric

Let A be a $m \times n$ matrix and $X = AA'$.

Now, since A' is the transpose of A, therefore A' is a $n \times m$ matrix.

Hence $X = AA'$ is a square matrix of order m.

Now, $X' = (AA')' = (A')'A' = AA'$; Art. 3.2 (i), (iv) $= X$

$\therefore$ X is symmetric by definition.

Ex. If $A = \begin{bmatrix} 1 & 1 \\ x & y \\ x^2 & y^2 \end{bmatrix}$**, find the value of** AA'**.**

Soln. Here $A' = \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \end{bmatrix}$

$$\therefore \quad AA' = \begin{bmatrix} 1 & 1 \\ x & y \\ x^2 & y^2 \end{bmatrix} \times \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot x + 1 \cdot y & 1 \cdot x^2 + 1 \cdot y^2 \\ x \cdot 1 + y \cdot 1 & x \cdot x + y \cdot y & x \cdot x^2 + y \cdot y^2 \\ x^2 \cdot 1 + y^2 \cdot 1 & x^2 \cdot x + y^2 \cdot y & x^2 \cdot x^2 + y^2 \cdot y^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & x + y & x^2 + y^2 \\ x + y & x^2 + y^2 & x^3 + y^3 \\ x^2 + y^2 & x^3 + y^3 & x^4 + y^4 \end{bmatrix}.$$

Note : Evidently AA' is a symmetric matrix.

II. If A **be any square matrix, then**

(a) $A + A'$ **is symmetric. (b)** $A - A'$ **is a skew-symmetric.**

Let $P = A + A'$ and $Q = A - A'$.

Then (a) $P' = (A + A')' = A' + (A')'$

$$= A' + A = A + A' = P.$$

$\therefore$ P is symmetric.

(b) $Q' = (A - A')' = A' - (A')'$

$= A' - A = -(A - A') = -Q.$

$\therefore$ Q is skew-symmetric.

Th. Any square matrix can be expressed uniquely as the sum of a symmetric and a skew-symmetric matrix.

[M:

Let A be a square matrix of order n and let

$$X = \frac{1}{2}(A + A') \text{ and } Y = \frac{1}{2}(A - A') \quad \ldots (1)$$

Then $X' = \frac{1}{2}(A' + A) = \frac{1}{2}(A + A') = X$

$\therefore$ X is symmetric.

and $Y' = \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A') = -Y.$

$\therefore$ Y is skew-symmetric.

Now from (1), $A = X + Y$ and hence the result follows.

To prove that the representation is unique, let $A = P + Q$ be another such representation of A, where P is symmetric and Q is skew-symmetric.

We want to show that $P = X$ and $Q = Y$.

We have $A' = (P + Q)' = P' + Q'$

$= P - Q;$ $\because P' = P$ and $Q' = -Q.$

$\therefore A + A' = 2P$ and $A - A' = 2Q.$

This $\Rightarrow P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A').$

Thus $P = X$ and $Q = Y$.

Therefore the representation is unique.

Ex. Express $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$

as the sum of a symmetric and a skew-symmetric matrix.

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Soln. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$ so that $A' = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$.

$$\therefore\ A + A' = \begin{bmatrix} 2 & 5 & 8 \\ 5 & 8 & 11 \\ 8 & 11 & 14 \end{bmatrix} \text{ and } A - A' = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

$$\text{Let } X = \frac{1}{2}(A + A') = \begin{bmatrix} 1 & 5/2 & 4 \\ 5/2 & 4 & 11/2 \\ 4 & 11/2 & 7 \end{bmatrix}$$

$$\text{and } Y = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & -1/2 & -1 \\ 1/2 & 0 & -1/2 \\ 1 & 1/2 & 0 \end{bmatrix}.$$

$\therefore A = X + Y$ where X is symmetric and Y is skew-symmetric.

IV. If A, B are symmetric matrices, then

(a) $A + B$ is symmetric

(b) AB is symmetric iff $AB = BA$

(c) $AB + BA$ is symmetric and $AB - BA$ is skew-symmetric.

Proof : Since A and B are symmetric matrices, we have, $A' = A$ and $B' = B$.

(a) Let $P = A + B$.

Then $P' = (A + B)' = A' + B' = A + B;\ \because A' = A$ and $B' = B$

$\Rightarrow\quad P' = P.$

$\therefore P$ is symmetric.

(b) Let $Q = AB$.

Then $Q' = (AB)' = B'A' = BA$ $(\because A' = A$ and $B' = B)$

$= AB$ if $AB = BA$

$\Rightarrow\quad Q' = Q.$

$\therefore Q$ is symmetric if $AB = BA$.

(c) Let $R = AB + BA$.

Then $R' = (AB + BA)' = (AB)' + (BA)'$

$= B'A' + A'B' = BA + AB = AB + BA = R.$

$\therefore R$ is symmetric.

Again, let $S = AB - BA$.

Then $S' = (AB - BA)' = (AB)' - (BA)'$

$= B'A' - A'B' = BA - AB = -(AB - BA) = -S.$

$\therefore S$ is skew-symmetric.